the equation of motion (1) for this case are $\alpha = 0.064203$, $\beta = -0.72285$, and $\gamma = 4.0822$. The frequency ratios ω/ω_L for the specified values of the amplitude ratios ξ_{max} are presented in Table 1. The hybrid-Galerkin method, assuming only two terms in the approximate displacement function (6), yields good results that are comparable with those obtained by exact integration.[11,13,14] The accuracy of the results can be further improved by assuming more numbers of terms in the approximate displacement function.

References

[1]Chia, C. Y., *Nonlinear Analysis of Plates,* McGraw-Hill, New York, 1980.

[2]Bert, C. W., "Research on Dynamics of Composite and Sandwich Plates," *Shock and Vibration Digest,* Vol. 14, No. 10, 1982, pp. 17-34.

[3]Satyamoorthy, M., "Nonlinear Vibrations of Plates—A Review," *Shock and Vibration Digest,* Vol. 15, No. 6, 1983, pp. 3-16.

[4]Satyamoorthy, M., "Nonlinear Vibrations Analysis of Plates," *Applied Mechanics Review,* Vol. 40, No. 11, 1987, pp. 1553-1561.

[5]Chandra, R., and Raju, B. B., "Large Deflection Vibration of Angle-Ply Laminated Plates," *Journal of Sound and Vibration,* Vol. 40, No. 3, 1975, pp. 393-408.

[6]Chandra, R., and Raju, B. B., "Large Amplitude Flexural Vibration of Cross-Ply Laminated Composite Plates," *Fibre Science and Technology,* Vol. 8, No. 4, 1975, pp. 243-264.

[7]Chia, C. Y., "Large Amplitude Vibrations of Laminated Rectangular Plates," *Fibre Science and Technology,* Vol. 17, No. 2, 1982, pp. 123-131.

[8]Bhimaraddi, A., "Nonlinear Free Vibration Analysis of Composite Plates with Initial Imperfections and In-Plane Loading," *International Journal of Solids and Structures,* Vol. 25, No. 1, 1989, pp. 33-43.

[9]Nageswara Rao, B., and Pillai, S. R. R., "Large Amplitude Free Vibrations of Laminated Anisotropic Thin Plates Based Harmonic Balance Method," *Journal of Sound and Vibration,* Vol. 154, No. 1, 1992, pp. 173-177.

[10]Singh, G., Kanakaraju, K., Venkateswara Rao, G., and Iyengar, N. G. R., "Nonlinear Vibrations of Simply Supported Rectangular Cross-Ply Plates," *Journal of Sound and Vibration,* Vol. 142, No. 2, 1990, pp. 213-226.

[11]Singh, G., Venkateswara Rao, G., and Iyengar, N. G. R., "Large Amplitude Free Vibration of Simply Supported Antisymmetric Cross-Ply Plates," *AIAA Journal,* Vol. 29, No. 5, 1991, pp. 784-790.

[12]Pillai, S. R. R., and Nageswara Rao, B., "Large Amplitude Vibration of Thin Plates," *Journal of Sound and Vibration,* Vol. 149, No. 3, 1991, pp. 509-512.

[13]Nageswara Rao, B., and Pillai, S. R. R., "Exact Solution of the Equation of Motion to Obtain Nonlinear Vibration Characteristics of Thin Plates," *Journal of Sound and Vibration,* Vol. 153, No. 1, 1992, pp. 168-170.

[14]Pillai, S. R. R., and Nageswara Rao, B., "Re-Investigation of Nonlinear Vibrations of Simply Supported Rectangular Cross-Ply Plates," *Journal of Sound and Vibration* (to be published).

[15]Pillai, S. R. R., and Nageswara Rao, B., "Improved Solution for Nonlinear Vibration of Simply Supported Rectangular Cross-Ply Plates," *Journal of Sound and Vibration,* Vol. 150, No. 3, 1991, pp. 517-519.

[16]Nageswara Rao, B., and Pillai, S. R. R., "Nonlinear Vibrations of a Simply Supported Rectangular Antisymmetric Cross-Ply Plates with Immovable Edges," *Journal of Sound and Vibration,* Vol. 152, No. 3, 1992, pp. 568-572.

[17]Reddy, J. N., and Chao, W. C., "Nonlinear Oscillations of Laminated Anisotropic Rectangular Plates," *Journal of Applied Mechanics, Transactions of the American Society of Mechanical Engineers,* Vol. 49, No. 2, 1982, pp. 396-402.

[18]Nayfeh, A. H., and Mook, D. T., *Nonlinear Oscillations,* Wiley-Interscience, New York, 1979.

[19]Mickens, R. E., *An Introduction to Nonlinear Oscillations,* Cambridge, Univ. Press, Cambridge, England, UK, 1981.

[20]Mickens, R. E, "Comments on the Method of Harmonic Balance," *Journal of Sound and Vibration,* Vol. 94, No. 3, 1984, pp. 456-460.

[21]Telban, R. J., Geer, J. F., and Pitarresi, M., "Hybrid-Galerkin Perturbation Method for Forced Oscillations of the Duffing Equation," *Mathematical and Computer Modelling,* Vol. 14, 1990, pp. 120-123.

Fully Nonlinear Model of Cables

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Introduction

Applications of cables include power cables, information transmission lines, towing and mooring marine vehicles, lines for tethering objects over long distances, uses in large buildings to provide large, column-free areas, tension members of cable-stayed bridges or guy-towers, guy cables for wind turbines, taut strings for musical instruments, etc. Because engineering cables are usually lengthy and flexible, their vibrations are essentially nonlinear and dominated by geometric nonlinearities. Large-amplitude vibrations and nonlinear phenomena of cables have been extensively studied analytically[1-11] and experimentally.[3,6,7] There are many nonlinear cable models due to the many ad hoc assumptions adopted by different researchers during different steps in the derivation. Because extensional forces are the only elastic loads acting on cables, compressibility and the Poisson effect can be significant. For example, for rubber-like materials the Poisson effect results in a nonlinear relationship[12] between the tension force and the axial strain. In the literature, most of cable models do not account for the change in the cross-sectional area due to the Poisson effect or assume that the material is incompressible; hence, the Poisson ratio ν is absent from the equations of motion. Such theories are valid only for materials with $\nu \simeq 0$ or $\nu \simeq 0.5$. But, the Poisson ratios of most engineering materials are between 0.25 and 0.35, except those of rubber and paraffin, for which $\nu \simeq 0.5$. Most researchers account only for quadratic or cubic nonlinearities. However, quadratic and cubic nonlinearities have different influences on nonlinear responses of cables[1,11]; hence, both of them need to be modeled. For large-amplitude vibrations, nonlinear terms of order higher than cubic are needed. For example, to study cable vibrations with one to five internal resonances, fourth- and fifth-order terms are needed.[1] Hence, we present a geometrically exact or fully nonlinear cable model, which can be expanded to any order, taking into account initial sags, static loads, and extensionality.

In this Note we use an energy approach to develop a fully nonlinear cable model, which contains most cable models as special cases. The theory accounts for the effects of static and dynamic loads, initial sags, compressibility, material nonuniformity, Poisson's effect, and geometric nonlinearities. Also, we derive cubic nonlinear equations of motion for sagged cables, taut strings, and extensional bars and compare the model with several other models.

Fully Nonlinear Equations of Motion

Figure 1a shows the deformed configuration of a cable and the inertial coordinate system x_1-x_2-x_3. Point P_0 indicates the position of the observed particle when the cable is not loaded, point $\hat{P}$ is the deformed position of P_0 under static loads, and point P is the deformed position of P_0 under static and dynamic loads. The coordinates of $\hat{P}$ are (α_1, α_2, α_3); u_1, u_2, and u_3 are the dynamic displacements along the axes x_1, x_2, and x_3, respectively; and the coordinates of P are (x_1, x_2, x_3). Thus,

$$x_i = \alpha_i + u_i \qquad \text{for} \qquad i = 1, 2, 3 \tag{1}$$

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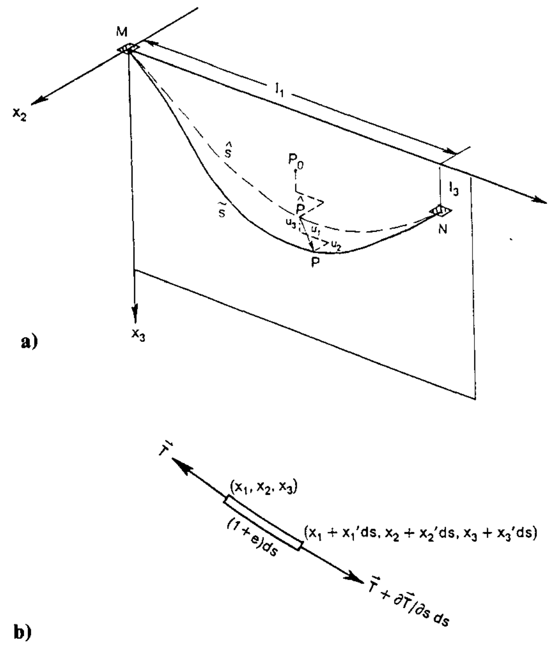

Fig. 1 Cable model: a) deformed configuration of a cable, where x_1-x_2-x_3 is a rectangular, inertial coordinate system; b) free-body diagram of an element.

In this Note s denotes the undeformed arc length measured from point M to the observed element; $\hat{s}$ represents the statically deformed arc length; $\tilde{s}$ represents the dynamically deformed arc length; L is the unloaded total length of the cable; $\rho_0(s)$ is the density of the unloaded cable; $A_0(s)$ is the cross-sectional area of the unloaded cable; and $(\ )' \equiv \partial(\ )/\partial s$.

It follows from Eq. (1) and the free-body diagram shown in Fig. 1b that the dynamic axial strain e is given by

$$e = \sqrt{(x_1')^2 + (x_2')^2 + (x_3')^2} - 1 = \sqrt{\eta_0 + 2\eta_1 + \eta_2} - 1 \tag{2}$$

where

$$\eta_0 \equiv (\alpha_1')^2 + (\alpha_2')^2 + (\alpha_3')^2$$

$$\eta_1 \equiv \alpha_1' u_1' + \alpha_2' u_2' + \alpha_3' u_3'$$

$$\eta_2 \equiv (u_1')^2 + (u_2')^2 + (u_3')^2 \tag{3}$$

Because of the Poisson effect, the deformed cross-sectional area A is given by

$$A = (1 - \nu e)^2 A_0 \tag{4}$$

where ν is the Poisson ratio. Using Hooke's law and Eq. (4), we obtain

$$T(s, t) = AEe(1 + E_1 e + E_2 e^2)$$

$$= EA_0(1 - \nu e)^2 e(1 + E_1 e + E_2 e^2) \tag{5}$$

where T denotes the internal tension force, $E(s)$ is Young's modulus, and $E_1(s)$ and $E_2(s)$ are constants due to material nonlinearity.

The extended Hamilton principle is used to derive the equations of motion, which can be stated as

$$0 = \int_0^t (\delta K - \delta V + \delta W_{nc})\, dt \tag{6}$$

Here, δK is the variation of the kinetic energy K and given by

$$\delta K = -\int_0^L m_0(\ddot{u}_1 \delta u_1 + \ddot{u}_2 \delta u_2 + \ddot{u}_3 \delta u_3)\, ds \tag{7a}$$

where $m_0 \equiv \rho_0 A_0$ and the overdot denotes differentiation with respect to time t. The variation δV of the elastic energy V can be obtained by using Eq. (2) as

$$\delta V = \int_0^L T\delta e\, ds = \int_0^L \left(\frac{Tx_1'}{1+e}\delta u_1' + \frac{Tx_2'}{1+e}\delta u_2' + \frac{Tx_3'}{1+e}\delta u_3' \right) ds \tag{7b}$$

Also, δW_{nc} denotes the variation of the nonconservative energy W_{nc} and is given by

$$\delta W_{nc} = \int_0^L (f_1 \delta u_1 + f_2 \delta u_2 + f_3 \delta u_3$$

$$- \mu_1 \dot{u}_1 \delta u_1 - \mu_2 \dot{u}_2 \delta u_2 - \mu_3 \dot{u}_3 \delta u_3)\, ds + \delta W_b \tag{7c}$$

where f_i denotes the total distributed static and dynamic load per unit of undeformed length along the x_i direction, and μ_i is the damping coefficient per unit of the undeformed length along the x_i direction. Moreover, δW_b is the virtual work due to forces applied on the boundaries or due to the motion of the boundaries, which is problem dependent and will not be considered in the derivation. Substituting Eqs. (7) into Eq. (6), integrating by parts, and then setting each of the coefficients of δu_1, δu_2, and δu_3 equal to zero, we obtain the equations of motion as

$$\frac{\partial}{\partial s}\left[\frac{T}{1+e}(\alpha_i' + u_i') \right] + f_i$$

$$= m_0 \ddot{u}_i + \mu_i \dot{u}_i \qquad \text{for} \qquad i = 1, 2, 3 \tag{8}$$

and the boundary conditions are of the following form: Specify

$$u_i \qquad \text{or} \qquad \frac{Tx_i'}{1+e} \qquad \text{at} \qquad s = 0$$

$$\text{and} \qquad L \qquad \text{for} \qquad i = 1, 2, 3 \tag{9}$$

Substituting Eqs. (5), (2), and (1) into Eqs. (8) and (9), one can obtain fully nonlinear equations of motion and boundary conditions in terms of α_i and u_i. To compare the model with others, we expand Eqs. (8) and (9) up to cubic terms in the following section.

Cubic Nonlinear Equations of Motion

Here, we assume that the material is linear and hence $E_1 = E_2 = 0$ in Eq. (5). It follows from Eqs. (8) that the static equilibrium states are given by

$$\frac{\partial}{\partial s}\left[\frac{\hat{T}}{1+\hat{e}}\alpha_i' \right] + \hat{f}_i = 0 \qquad \text{for} \qquad i = 1, 2, 3 \tag{10}$$

and the boundary conditions: Specify

$$\alpha_i \qquad \text{or} \qquad \frac{\hat{T}}{1+\hat{e}}\alpha_i' \qquad \text{at} \qquad s = 0$$

$$\text{and} \qquad L \qquad \text{for} \qquad i = 1, 2, 3 \tag{11}$$

where $\hat{T} = EA_0(1 - \nu\hat{e})^2\hat{e}$, the $\hat{f}_i$ denote the static loads along the x_i axes, and $\hat{e}$ is the strain due to static loads, which is obtained from Eq. (2) as

$$\hat{e} = \sqrt{\alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2} - 1 \tag{12}$$

We subtract Eqs. (10) and (11) from Eqs. (8) and (9), expand the result up to cubic terms, and obtain the following equations of motion:

$$m_0 \ddot{u}_i + \mu_i \dot{u}_i = \tilde{f}_i + (A_0 E U_i)' \qquad \text{for} \qquad i = 1, 2, 3 \tag{13}$$

and the following expanded boundary conditions: Specify

$$u_i \quad \text{or} \quad A_0 E U_i \quad \text{at} \quad s = 0 \quad \text{and} \quad L \qquad (14)$$

for $i = 1, 2, 3$, where the dynamic loads $\tilde{f}_i \equiv f_i - \hat{f}_i$, and

$$\begin{aligned} U_i &\equiv \Lambda_1 u_i' + \Lambda_2(\alpha_i' + u_i')\eta_1 + \Lambda_3(\alpha_i' + u_i')\eta_1^2 \\ &\quad + \tfrac{1}{2}\Lambda_2(\alpha_i' + u_i')\eta_2 + \Lambda_3\alpha_i'\eta_1\eta_2 + \Lambda_4\alpha_i'\eta_1^3 \\ \Lambda_1 &\equiv (1 + \nu - \nu\eta_0^{1/2})^2(1 - \eta_0^{-1/2}) \\ \Lambda_2 &\equiv 2\nu^2 - \eta_0^{-1/2}(2\nu + 3\nu^2) + \eta_0^{-3/2}(1+\nu)^2 \\ \Lambda_3 &\equiv \eta_0^{-3/2}[\nu + (3/2)\nu^2] - (3/2)\eta_0^{-5/2}(1+\nu)^2 \\ \Lambda_4 &\equiv -\eta_0^{-5/2}[\nu + (3/2)\nu^2] + (5/2)\eta_0^{-7/2}(1+\nu)^2 \end{aligned} \qquad (15)$$

We note that the $\Lambda_2\alpha_i'\eta_1$ are linear coupling terms due to the initial sag, and there are no such couplings if two of the α_i' are equal to zero, corresponding to the case of taut strings. The nonlinear terms in Eqs. (13) and (14) are due to geometric nonlinearities and the Poisson effect. The quadratic terms in Eq. (13) are the same as those of Tadjbakhsh and Wang,[5] but the coefficients are different because they considered incompressible materials.

Taut Strings

If there is no initial sag and the string is placed along the x_1 direction, then

$$\alpha_1' = 1 + \hat{e}, \qquad \alpha_2' = \alpha_3' = 0, \qquad \eta_0 = \alpha_1'^2, \qquad \eta_1 = \alpha_1' u_1' \qquad (16)$$

Substituting Eqs. (16) into Eqs. (15) yields

$$\begin{aligned} U_1 &\equiv \hat{\Lambda}_1 u_1' + \hat{\Lambda}_2 u_1'^2 + \nu^2 u_1'^3 + (\tfrac{1}{2}\Lambda_2\alpha_1' + \hat{\Lambda}_3 u_1')(u_2'^2 + u_3'^2) \\ U_2 &\equiv \Lambda_1 u_2' + \Lambda_2\alpha_1' u_1' u_2' + \hat{\Lambda}_3 u_1'^2 u_2' + \tfrac{1}{2}\Lambda_2 u_2'(u_2'^2 + u_3'^2) \\ U_3 &\equiv \Lambda_1 u_3' + \Lambda_2\alpha_1' u_1' u_3' + \hat{\Lambda}_3 u_1'^2 u_3' + \tfrac{1}{2}\Lambda_2 u_3'(u_2'^2 + u_3'^2) \end{aligned} \qquad (17)$$

where

$$\begin{aligned} \hat{\Lambda}_1 &\equiv 1 + 4\nu + 3\nu^2 - 2\nu(2 + 3\nu)\alpha_1' + 3\alpha_1'^2\nu^2 \\ \hat{\Lambda}_2 &\equiv 3\nu^2\alpha_1' - 2\nu - 3\nu^2 \\ \hat{\Lambda}_3 &\equiv \nu^2 - (1+\nu)^2/\alpha_1'^3 \end{aligned} \qquad (18)$$

Neglecting the Poisson effect [i.e., setting $\nu = 0$ in Eqs. (15), (17), and (18)], using the identity $d\hat{s} = (1 + \hat{e})\, ds = \alpha_1'\, ds$, denoting $\partial(\ )/\partial\hat{s}$ by $(\ )^+$, and assuming that m_0, A_0E, and α_1' are constants, we rewrite Eqs. (13) as

$$\ddot{u}_1 + \frac{\mu_1}{m_0}\dot{u}_1 - K_2^2 u_1^{++} = \tilde{f}_1 + K_1^2\left[\left(\frac{1}{2} - u_1^+\right)\left(u_2^{+2} + u_3^{+2}\right)\right]^+ \qquad (19a)$$

$$\ddot{u}_2 + \frac{\mu_2}{m_0}\dot{u}_2 - (K_2^2 - K_1^2)u_2^{++} = \tilde{f}_2 + K_1^2\left[u_2^+\left(u_1^+ - u_1^{+2} + \frac{1}{2}u_2^{+2} + \frac{1}{2}u_3^{+2}\right)\right]^+ \qquad (19b)$$

$$\ddot{u}_3 + \frac{\mu_3}{m_0}\dot{u}_3 - (K_2^2 - K_1^2)u_3^{++} = \tilde{f}_3 + K_1^2\left[u_3^+\left(u_1^+ - u_1^{+2} + \frac{1}{2}u_2^{+2} + \frac{1}{2}u_3^{+2}\right)\right]^+ \qquad (19c)$$

where $K_1^2 \equiv EA_0\alpha_1'/m_0$ and $K_2^2 \equiv EA_0\alpha_1'^2/m_0$. We note that the terms $u_1'^2$ and $u_1'^3$ are missing in Eq. (19a) because they are due to the Poisson effect. Equations (19) are the same as Eqs. (7.5.9–7.5.11) of Nayfeh and Mook,[1] but the coefficients are slightly different. The reason for these differences is that we neglected the Poisson effect in both the static and dynamic deflections, but Nayfeh and Mook neglected the Poisson effect only in the dynamic deflections; and we defined the strain with respect to the undeformed length s, but Nayfeh and Mook used the deformed length $\hat{s}$. All quadratic terms in Eqs. (19) are missing in the model used in Ref. 10.

Extensional Bars

If there are no initial sags or static loads, the string is placed along the x_1 axis, and $u_2 = u_3 = 0$. Then

$$\alpha_1' = 1, \qquad \alpha_2' = \alpha_3' = 0, \qquad \eta_0 = 1, \qquad \eta_1 = u_1', \qquad \eta_2 = u_1'^2 \qquad (20)$$

Consequently, Eqs. (13) for $i = 1$ reduce to

$$m_0\ddot{u}_1 + \mu_1\dot{u}_1 = f_1 + [A_0E(u_1' - 2\nu u_1'^2 + \nu^2 u_1'^3)]' \qquad (21)$$

and the boundary conditions: Specify

$$u_1 \quad \text{or} \quad A_0E(u_1' - 2\nu u_1'^2 + \nu^2 u_1'^3) \quad \text{at} \quad s = 0 \quad \text{and} \quad L \qquad (22)$$

Equations (21) and (22) represent the equation of motion and boundary conditions for an extensional bar without static loads. For extensional bars there are no geometric nonlinearities involved, and the nonlinear terms in Eqs. (21) and (22) are due to the Poisson effect. We note that nonlinearities due to the Poisson effect can be mistaken for material nonlinearities.

Method for Nonlinear Static Solutions

Considering the system shown in Fig. 1a and assuming that the material is incompressible and linear, we obtain the following from Eqs. (10), (12), and (5):

$$\alpha_i = \int_0^s \frac{A_0E(F_i - g_i)}{A_0E\hat{T} - \hat{T}^2}\, ds \qquad \text{for} \qquad i = 1, 2, 3 \qquad (23)$$

where

$$g_i \equiv \int_0^s \hat{f}_i\, ds$$

$$F_i \equiv \frac{\hat{T}}{1 + \hat{e}}\alpha_i' \,|_{s=0}$$

$$\hat{T} = \sqrt{(F_1 - g_1)^2 + (F_2 - g_2)^2 + (F_3 - g_3)^2}$$

Because g_i, A_0, and E are functions of s and F_i are constants, Eq. (23) can be integrated to obtain α_i. However, the F_i are unknown constants and need to be determined by matching the integral solutions with the boundary conditions at $s = L$ (i.e., $\alpha_1 = l_1$, $\alpha_2 = 0$, $\alpha_3 = l_3$). A shooting method can be used to solve this nonlinear two-point boundary-value problem.

Closure

The present three-dimensional nonlinear theory for cables fully accounts for geometric nonlinearities, nonlinearities due to the Poisson effect, compressibility, nonuniformity, initial sags, and static and dynamic loads. The nonlinear equations of motion show linear couplings due to initial sags and static loads and nonlinear couplings due to large deflections and the Poisson effect. Also, cubic nonlinear equations of motion for sagged cables, taut strings, and extensional bars are developed, and comparisons with other derivations are made. Moreover, the possibility of including material nonlinearities in the model is addressed.

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References

[1]Nayfeh, A. H., and Mook, D. T., *Nonlinear Oscillations,* Wiley-Interscience, New York, 1979, Chap. 7.

[2]Beatty, M. F., and Chow, A. C., "Free Vibrations of a Loaded Rubber String," *International Journal of Non-Linear Mechanics,* Vol. 19, No. 1, 1984, pp. 69–82.

[3]Pastorel, H., and Beaulieu, G., "Nonlinear Vibrations of Guy Cable Systems," *Computers and Structures,* Vol. 21, No. 1/2, 1985, pp. 33–50.

[4]Ali, S. A., "Dynamic Responses of Sagged Cables," *Computers and Structures,* Vol. 23, No. 1, 1986, pp. 51–57.

[5]Tadjbakhsh, I. G., and Wang, Y. M., "Wind-Driven Nonlinear Oscillations of Cables," *Nonlinear Dynamics,* Vol. 1, No. 4, 1990, pp. 265–291.

[6]Fujino, Y., Warnitchai, P., and Pacheco, B. M., "An Experimental and Analytical Study of Autoparametric Resonance in a 3dof Model of Cable-Stayed-Beam," *Nonlinear Dynamics* (to be published).

[7]Papazoglou, V. J., and Mavrakos, S. A., "Nonlinear Cable Response and Model Testing in Water," *Journal of Sound and Vibration,* Vol. 140, No. 1, 1990, pp. 103–115.

[8]Mitsugi, J., and Yasaka, T., "Nonlinear Static and Dynamic Analysis Method of Cable Structures," *AIAA Journal,* Vol. 29, No. 1, 1991, pp. 150–152.

[9]Rao, G. V., and Iyengar, R. N, "Internal Resonance and Nonlinear Response of a Cable under Periodic Excitation," *Journal of Sound and Vibration,* Vol. 149, No. 1, 1991, pp. 25–41.

[10]O'Reilly, O., and Holmes, P., "Nonlinear, Nonplanar and Nonperiodic Vibrations of a String," *Journal of Sound and Vibration* (to be published).

[11]Lee, C. L., and Perkins, N. C., "Nonlinear Oscillations of Suspended Cables Containing a Two-to-One Internal Resonance," *Nonlinear Dynamics* (to be published).

[12]King, A. L., "Oscillations of a Loaded Rubber Band," *American Journal of Physics,* Vol. 42, No. 8, 1974, pp. 699–701.

Beam Deflection

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I. Introduction

THERE have been numerous attempts to improve the classical Euler-Bernoulli beam theory, not to mention the work by Timoshenko.[1] Following Timoshenko, a number of high-order engineering beam theories have been proposed, including works by Duva and Simmonds,[2] Fan and Widera,[3] Rehfield and Murthy,[4] and Rychter.[5] From the viewpoint of the theory of elasticity, Gregory and Wan[6] derive the exact condition that the outer beam solutions must satisfy for decaying states, and Wan[7] offers a correct beam theory. On the other hand, Gregory and Gladwell,[8] Adams and Bogy,[9] and Robert and Keer[10] use the elasticity solution to study stresses, including the edge effects in clamped beams. However, either higher order theories or analyses based on the elasticity theory do not appear simple enough to be used in daily applications. The purpose of this Note is to indicate that a simple modification in the beam theory predicts beam deflections much better than does the Euler-Bernoulli beam theory. The deflections obtained by this modification agree very well with those by the *plate* finite elements. This modification is particularly useful in beams with some kinematic boundary conditions, such as clamped or simply supported. We remark that even in very narrow beams the deflections by the classical Euler-Bernoulli beam theory do not agree very well with those by the plate finite element analysis.

For the modification, we start from the classical plate theory and try to impose geometric constraints from the beginning of derivation. Although beams with kinematic end constraints such as simple or clamped supports are widely used in engineering applications, it appears that in the literature there is no attempt to take explicitly into account such constraints. Since the Euler-Bernoulli theory is derived for beams under pure bending (which are not constrained) and directly extended for beams under other loading and boundary conditions, it can be expected that the theory may not distinguish the different behavior of beams with and without kinematic constraints. Although the result obtained by the present modification is exactly the same as that for wide beams (e.g., see Budynass[11]), the derivation procedures are fundamentally different. It is shown that the present modification can be used for a wide range of beam widths, from very narrow to relatively wide.

II. Analysis for the Proposed Modification

For later use, we just list the Euler-Bernoulli theory for a beam with rectangular cross section (e.g., see Crandall et al.[12]):

$$\frac{d^2 M_x}{dx^2} = -p(x) \tag{1}$$

$$M_x = EI\kappa_x \tag{2}$$

$$\kappa_x = -\frac{d^2 w}{dx^2} \tag{3}$$

$$EI = \frac{Et^3}{12} \tag{4}$$

The transverse load and deflection are denoted by $p(x)$ and $w(x)$, respectively, and x is the axial coordinate located in the middle plane of a beam. The moment M_x is measured per unit width, and $p(x)$ has the dimension of pressure. E and t represent Young's modulus and the beam thickness. Note that the beam theory is based on the plane-stress assumption in the direction perpendicular to the plane.

Unlike in the derivation of the classical beam theory where pure bending is assumed, we rather start from the Kirchhoff-Love plate theory and consider the effect of kinematic constraints from the beginning of the derivation. The well-known Kirchhoff-Love plate theory is briefly reviewed here (e.g., see Ugural[13]):

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p(x,y) \tag{5}$$

$$M_x = D(\kappa_x + \nu\kappa_y) \tag{6a}$$

$$M_y = D(\kappa_y + \nu\kappa_x) \tag{6b}$$

$$M_{xy} = D(1-\nu)\kappa_{xy} \tag{6c}$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \tag{7a}$$

$$\kappa_y = -\frac{\partial^2 w}{\partial y^2} \tag{7b}$$

$$\kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \tag{7c}$$

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{8}$$

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